POWER SERIES IN FRANCE DURING LATE XIXTH CENTURY

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Abstract. This article is one spin-off from [1]. We revisit the most relevant developments in the theory of complex power series during late XIX$^{\text{th}}$ in France.

We will comparatively show that the later investigation on the dynamics of functions in one complex variable over the Riemann sphere was a natural consequence from the crucial events here narrated.

Est modus in rebus: sunt certi denique fines,
quos ultra citraque nequit consistere rectum. †

Oratius,
from *Satire* (1, 1, 106-107).

1. Spoiling the end

In 1910 Paul Montel (1876–1975) published a memoir on the series of polynomials [70]. Among several interesting results, we focus on these interesting excerpts:

> (p. 100) "Thus the singularities of a function cannot belong to the region of convergence of a series of polynomials representing the function itself".
>
> (p. 100) "It comes that some points where the function is regular do not belong to the domain of convergence of the series itself [. . .] It is clear that the existence of boundary curves is indispensable around the critical points."
>
> (p. 119) "Given a series of polynomials, converging into a domain D, the set of irregular points is perfect, non dense in D, continuous and connected with the domain boundary."

Which is the connection between complex power series and the iterations of complex functions ?

2. An introduction to series

Series flourished in Europe during XIXth century as multi-purpose tool to attack different questions, such as primes distribution, approximation of polynomial roots, differential equations. In 1850s we count three distinguished examples: Dirichlet's, Fourier's and Taylor's, tagged according to their discoverers or major developers.

Key words and phrases. Taylor series, Power series, continuation, complex analysis, history.

† '*There is a measure in things: there are precise confinements, over which and before which the right cannot subsist.*'

Dirichlet series are of the form

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}.$$

Since Euler, who worked on the case of $a_n = 1, s \in \mathbb{R}$, they immediately found applications in Number Theory. Later in 1830s, Mobius ([69], 1832) and especially Dirichlet ([28], 1834) made important broadenings for $a_n, s \in \mathbb{C}$. But the most famous example

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s},$$

where $a_n = 1, s \in \mathbb{C}$, is due to Riemann [89]: this is the so-called 'the Riemann's zeta function', essential to investigate primes distribution. More distinguished results for Dirichlet series followed by Cahen, de la Vallée-Poussin, Hadamard, Landau, Lindelöf, Perron, Riesz, von Mangoldt.

A second example is Fourier's, belonging to the wider family of trigonometric series:

$$f(x) : \lim_{r\uparrow 1}\sum_{0}^{\infty} r^n(a_n\cos nt + b_n\sin nt) \ \textit{where}\ f(x) = \begin{cases} 1, 0 \le x < \pi, \\ -1, \pi \le x < 2. \end{cases} \tag{2.1}$$

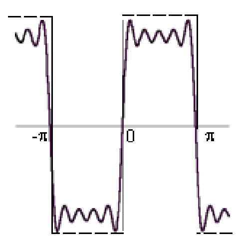

Figure 1. A periodic $f(x)$ represented by Fourier series.

During 1870's and early 1880's, vivid studies were nourished by Arzelà, Ascoli, Dini (in *Italy*), by Appell, Bonnet, Boussinesq, Darboux, Poincarè (in *France*), by Cantor, du-Bois Reymond, Hankel, Hermite, Holder, Kronecker, Pringsheim, Weierstrass (in *Germany*). But they also found some hostility among the gotha of French analysts in late 80's: for example, Hermite and Poincarè [51] saw them as hardly tamed beasts due to the possibility of constructing *continuous but nowhere differentiable functions*. A scandal in those times.

We finally come to the third example: the Taylor series[1]

$$T(z) = \sum_{n=0}^{\infty} c_n(z-a)^n, c_n = \frac{f^{(n)}(a)}{n!}. \tag{2.2}$$

Let $z \in \mathbb{C}$. This is the power series expansion of an analytic map $f(z)$ about a point a (*algebraic* definition: see [71], p. 104) and $f^{(n)}$ are the successive derivatives of the function f. Their major application is indeed to locally approximate functions. Taylor's belong to the wider family of Laurent series that extend to pole-like singularities:

$$L(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n; \tag{2.3}$$

if $n \ge 0$, we get back to (2.2). Terms summation could tend to bounded values (*convergence*) or to infinity (*divergence*). In this dichotomy, series 'succeed' or 'fail' to represent $f(z)$ and a is tagged as *regular point* or *singularity* respectively. Convergence may thus not hold anywhere. To approximation goals, this is a very big issue, so it is crucial to ask: where does $T(z)$ converge? Where is it defined?

[1] A *function element* is an ordered pair (f, U), where U is a disc $D(P, r)$ and f is a holomorphic map defined on U (*geometrical* definition, see [42] p. 304).

Where is it not? Values cannot be picked up at random: to prevent serious risks in real world, Engineers and Physicists want mathematical models keeping error probability as close as we please to 0. *Applications do not leave much room to divagations.*

3. Turn the power(s) on

Series started as a tool for algebraists in Real analysis. When coefficients turned to complex numbers - as their definitive generalization, the exploration broke into Geometry too. The leaders of this '*explo*-sion' were Augustine Cauchy (1789–1857) and Karl Weierstrass (1815–1897): they pioneered two independent approaches to complex holomorphic functions and coined the attributes '*monogenic*'[2] and '*analytic*'[3] respectively.

Let F be a function in one complex variable. During 1821–26, Cauchy developed two integral-based expressions

$$\oint_S F(z)dz = 0, \tag{3.1}$$

$$F^n(a) = \frac{n!}{2\pi i}\oint_S \frac{F(z)}{(z-a)^{n+1}}dz \tag{3.2}$$

Known as Cauchy's integral *theorem* and *formula*, they build a basic tools set to test the holomorphy of $F(z)$ inside a given subset $D \subset \mathbb{C}$ (3.1) and to compute derivatives of any order (3.2). Cauchy supposed ∂D to be a closed path S and showed that $F(z)$ can be approximated by a Taylor series all over S. If D is maximal, it is said the 'domain K of convergence' for (2.2) and the boundary ∂K includes singularities, where (2.2) does not converge.

This new concept opens to the question on the maximality of D: it is quickly exhausted for Taylor series, where the theory shows that K is disc-shaped and singularities spread over the bounding circle ('circle of convergence'). It makes sense to compute the extension of K via its radius r_K. Cauchy founds ways through the coefficients of (2.2):

$$r_K = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}} \tag{3.3}$$

Thus Cauchy got the light. Let's see about making a torch.

Cauchy's inheritance was weakly taken on by his countrymen: up to 1870's, several French [4] took up the study of complex Taylor series, but no results are

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[2] Having only a single derivative at each point.

[3] Having a convergent power series expansion at each point.

[4] We mention Bonnet, Bouquet, Briot, Lamarle, Marie, Méray, Puiseux ([67], p. 53). The latter was known for his homonymous series, extending Taylor series to rational exponents:

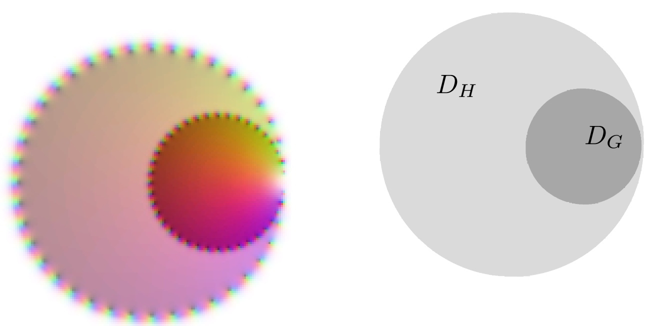
$$\sum_{n=p}^{\infty} \alpha_n z^{n/q}, \qquad p, q \in \mathbb{Z}, \qquad q \geq 1.$$

Marie had already worked on real Taylor series during 1860's but, as he tried to move to complex analogues in early 1870's, he lamented greater difficulties ([66], p. 469).

worth mentioning. Maximilien Marie blamed[5] French analysts for their lack of rigor and interest to questions related to Taylor series ([67], p. 53).

In early 1870s, the French left open questions such as determination of the domain of convergence via coefficients, the role of the critical points, the rectification of the perimeter of the polygon of convergence. French analysis was weak and also lagging behind the Germans at that time (or perhaps were not aware of the full extent of what the Germans were doing): to our goals, one indication right came from Koenigs's work (1884) on the local behavior of iterates of complex holomorphic functions ([53], p. 287), where two Darboux's theorems – on the uniform convergence of series of real functions to complex functions – were extended, ignoring that Weierstrass already did it four years earlier [2]. Weierstrass' work [94] was translated [95] in 1879 by Picard, three years after its publication in Germany, probably to support the circulation of these concepts in France.

In Germany, Weierstrass had in fact developed a *general theory of the analytic functions* as part of his larger and ambitious plan to rigorously systematize the whole corpus of Analysis. Among his most clever contributions was the '*analytic continuation*',[6] a method that runs locally by scanning K through a disk C, resizable as required and moved until ∂K is met.[7]



(A) (B)

FIGURE 2. The disc D_G is the domain of convergence for $G(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ and D_H for $H(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n\right]$, so that $D_H \supset D_G$. Since $H(z)$ assumes the same values in D_G exactly as $G(z)$ does, $H(z)$ is said the *analytic continuation* of $G(z)$. (B) resumes the super-imposition diagram in (A).

[5]More straight accusations will be also addressed by Gaston Darboux some years later [2, 39].

[6]*Analytic continuation*: given a connected open set $V \subset \mathbb{C}$, and two maps $f(z) : V \to \mathbb{C}$, $g(z) : V \to \mathbb{C}$, if $U \subset V$ and $f(U) \equiv g(U)$, then $f \equiv g$ all over V: *see* fig. 4 (A). The extension of f to g is unique. So let $M \subset \mathbb{C}$, $N \subset \mathbb{C}$ be open sets. If $f(M \cap N) \equiv g(M \cap N)$, provided $M \cap N \neq \emptyset$, then g is the analytic continuation of f. *See* [42], p. 304 or [36], p. 234 and fig. 4 (B) for a related application. Starting from a function f, the goal of continuation is to find *progressively* (i.e., attempt by attempt) the function g extending maximally the domain U. Thus, given $f \equiv g$, Weierstrass was able to infer that the concept of *analytic function* was to be differed from that of *analytic expression*.

[7]In [97] (p. 1), Cauchy was regarded as the true father of the theory for the above mentions.

Meditation The question on the geometries assumed by the domain of convergence has a straightforward response only for Taylor series. More complicate formulas demand new tools which just evince the properties, not its exact shape.

4. And the French stepped up the game

From late 1880's, new generations of researchers rescued the precarious state of French Analysis and lifted it to revolutionary scenarios. At no cheap cost anyway: these new results sparkled passionate controversies. In the next sections, we will review some events that aid to get a full picture of what was all about series and how they can be related to complex dynamics.

Following Zoretti[8], we start from 1887: under the advisorship of Émile Picard, the young Paul Painlevé (1863–1933), aged 24, published his thesis [72] on singular lines L of analytic functions $F(z)$. In a close neighborhood of L, analytic continuation becomes questionable, opening to the chance that F could exist beyond L, where its formula represents a new function $F_2(z)$. The split between functions and their formal expressions was straight ([72], p. 28):

Proposition A necessary and sufficient condition for $F(z)$, defined on a side C of a cut L and holomorphic in the neighborhood of L, to be continuable over L is the existence of a function $\varphi(z)$ defined on the opposite side of L, uniform in the neighborhood of L and taking on the same value as $F(z)$ at each point of L (with the exception of a set of points).[9]

The cut in question was assumed here to be an *analytic* line[10]. If $F(z)$ can be continued L, the latter is said to be *artificial*, otherwise it is *essential*[11] (p. 19). Apparently it looks like a tautology. Reviewing the passage '*at each point of the line*', we have $f(z) \equiv \varphi(z)$ over L, which holds all over S so that $\varphi(z)$ is the continuation of $F(z)$; Painlevé showed that the configuration applies to *arbitrarily shaped curves* (p. 29). And we get the sketch in fig. 3: two complex series taking on the same values at infinitely many points of L.

Painlevé classified (p. 67) cuts according to nearby singularities, opened to the chance that the boundary could *consist of analytic lines* (see fig. 3/A) and that essential cuts could include zeros, poles or essential singularities (p. 71).

Picard's report [78] of the thesis accounts that 4 opened to interesting deductions if $F(z)$ was no longer holomorphic on L, as well as to help to find conditions for a map to be analytically continued beyond a curve.

Meditation Fig. 3/B barely sketches the Julia sets action: *curves of singularities assumed by distinct functions, being defined at opposite sides.*

In the same year, Lecornu, aged 33, worked [58] on on the positions of the critical[12] points over the boundary of the domain of convergence and solved the question when one such point is on the boundary. This work inspired Hadamard (1865–1963), aged 23 in 1888, to ambitiously embark on Taylor series ([44], p.

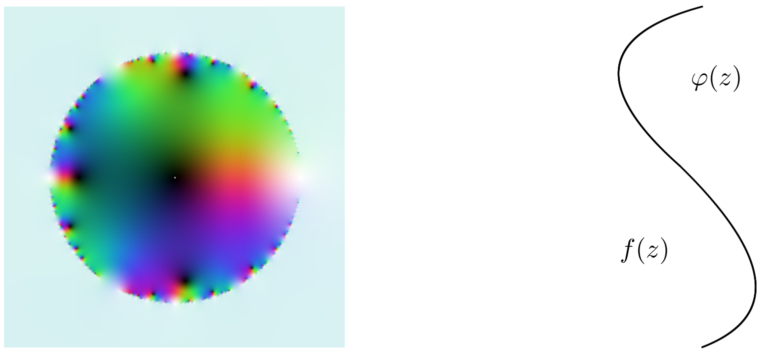
[8]Read [97], p. 5: Painlevé's thesis was among the earliest attempts to make clear the '*somewhat complicate Weierstrass' definition*'.

[9]We took the liberty of changing the original 'AB' to our 'L'.

[10]$F(z)$ can be represented over AB again as through convergent series.

[11]Belonging to the essence of its formula, in computational terms.

[12]This question is relevant for localizing extremum points, such as minima or maxima and brings a meaningful insight to the extension of the domain of convergence. Critical points also related to regions of non-invertibility.



(A) (B)

FIGURE 3. (A) for the series $\sum \frac{z^n}{1-z^n}$, the boundary consists of cuts which spread over disjoint arcs of the unit circle. (B) A common line shared by the domains of convergence of two complex functions, according to Painlevé's statement. It can be reviewed, via iterations, as $f(z) \equiv \phi^n(z)$ and $g(z) \equiv \rho^n(z)$.

259)[13]. It was known that if there exist non-singular points on the circle of convergence L of a series $T(z)$, the formula F defines a new series, deduced from $T(z)$ and defined at points beyond L. In 1890 Fredholm (1866–1927) gave an example [38] of Taylor series where L is a singular line and continuation cannot extend beyond it (fig. 4/A):

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{(n^2)}}{2^n}. \tag{4.1}$$

In fact L has the power of continuum, due to $2^{\aleph_0} > \aleph_0$ (where $\aleph_0$ is the cardinality of $\mathbb{N}$): no path, led by $F(z)$ from the interior of L, can trespass it.

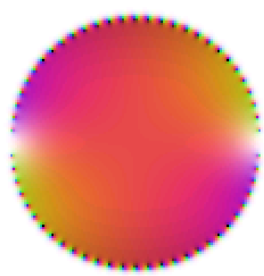


FIGURE 4. Disc D of convergence for series (4.1): boundary points are infinitely many and obstruct the analytic continuation to extend over ∂D.

[13]"One could determine, if existing, its circle of convergence. This question has been discussed by Lecornu in the case where the modulus $\dfrac{a_{m+1}}{a_m}$ or of $\sqrt[m]{a_m}$ has a limit in which case the limit is the inverse of the radius of convergence. The object of my present work is to solve the question for all cases."

In 1892, Poincaré gave an example of '*lacunary space*', a two-dimensional region bounded by an essential (alternatively said, natural). Relying on growth theory, he built a special configuration of essential cuts allowing continuation over them (see fig. 5 and refer to p. 213 of [80]):

" 1°: Both f and f_1 exist all over $\mathbb{C}$, except along given cuts.
2°: $\phi + \psi = f$ exists in the upper half-plane.
3°: $\phi + \psi = f_1$ exists in the lower half-plane.
4°: The cut of ϕ is the real segment between the points $x = \pm 1$.
5°: The cut of ψ consists of two real segments $(-\infty, -1)$ and $(+1, +\infty)$."

So if F can be continued in the lower half-plane, then F_1 is the *natural continuation* of F.

Meditation This outstanding example shows that complex functions composition can give birth to new relations, whose properties are unexpected and not enjoyed by their 'generators'.

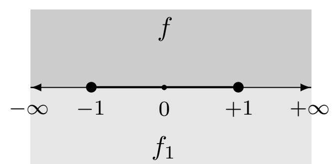


FIGURE 5. Poincaré's example of lacunary spaces. Each map, f or f_1, is defined inside one region but not outside, according to the two shades of grey.

In 1892 Hadamard presented his thesis [45]. The goal was to give a formula of the radius of the circle of convergence for general entire series[14]. This works consists of three parts: the first focuses on the convergence radius, the second and the third ones treat discontinuity for entire series around bounding pole-like singularities of any order. It was warmly welcomed by the French mathematical community for its avant-garde character, suggesting new, promising directions. Out of the choir, Picard lamented that the work fell short of its lofty goals anyway: the discussion often floats over too high levels of generalization so that its formality prevents to find out applications [79]. A prelude to the further clash between Theory and Practice.

5. The break in continuation

We place at the begin of 1890s. Analysis and Geometry were dragging the caravan of Mathematics across new arduous paths, paved by serious challenges that upset this Science to its foundations.[15] Hadamard has already rang the alarm: the faith into old concepts was doomed to fall !

There was no much left from the original Taylor series: Hadamard himself detached coefficients from derivatives, Painlevè showed new cases where analytic continuation did no longer fit to the cause. So we also drop the wording '*Taylor series*', except where strictly required, in favor of the more general form '*power series*'.

[14]Formulas where the denominator of each term does not include the variable.

[15]See [26], discussing on the contents of the deep correspondence between Borel-Hadamard-Zermelo about Set Theory in the same years.

In 1894 a young protagonist, Émile Borel (1871–1956), aged 23, emerged: his thesis, entitled *Sur quelques points de la théorie des fonctions*, focused on the rational series

$$F(z) = \sum \frac{A_n}{(z-a)^n}; \tag{5.1}$$

they are Laurent-like, possibly including poles. Borel worked on the properties of (5.1), manipulating the convergence speed of $\sum A_n$. Similarly to Poincaré's work on functions with lacunary spaces (p. 7 here), he set up two disjoint regions in $\mathbb{C}$, each one related to a distinct function. Then he linked his results to the new Cantor's set theory. What a bravery, if we assume that the latter had not much credit in its early years! And in a very short span of time, Borel became an authority in Analysis. Let's see how.

Weierstrass' analytic continuation was working out so fine for Taylor series, whose domain D of convergence is a disc and the singularities distribution enjoys some regularity. This is a local approach that uses a small disc C to scan the neighborhood for singularities and boils down to a number of attempts in all directions, until one of such special points is met. The latest discoveries urged a more careful approach, running through privileged directions. During late 1890s, Mittag-Leffler developed a new continuation approach using a star-like (*étoile*) set S ([68], 1st note, p. 47–48): given the series f approximating a function F, the general construction sets up one star branch $\overline{AB}$ of S and a point $C \in \overline{AB}$ so that C is not a singularity for f. Then $f(C)$ maps to another point $C_1 \in \overline{AB}$, assuming the uniform convergence of f; the branch $\overline{AB}$ could be a straight segment or a curve (p. 48). Today it is known as *Mittag-Leffler star*.

It seemed that Mittag-Leffler did the trick ! In contrast to the circle, the segment let the continuation earn lots of *flexibility*. Disks became too 'bulky' and such a thin line could snake across any distribution of singularities. Borel had also developed one similar method and, between 1899–1902, wrote some Comptes Rendus notes, remarking that it *did not replace, but just extended old standards*. Borel paid some reverence to the establishment in Analysis: in several instances, he remarked that this was not a revolution against Weierstrass' theory, but just an extension. Actually, this new idea made both followers and opponents. In 1900, the star-like continuation was an 'adaptive' method presenting these features:

1. f is not required to be analytic ([18], p. 1116);
2. the radial distribution of branches needs not to be the same all round ([18], p. 1117) ;
3. the star S could also work around a singular point γ, whatever γ is isolated or it belongs to a cut, because it could be 'jumped over' and skipped along one branch of S ([19], p. 151);
4. rectilinear segments can be replaced by not self-intersecting arcs; for example, spiral-like branches ([19], p. 151);
5. branches are independent from each other: if a singularity is met, one branch can be shortened;
6. given a sequence of points $x_n \to x_0$, all over the branch, the sequence of $f(x_n)$ converges uniformly to $f(x_0) \equiv F(x_0)$.

The point 3 opened to the existence of functions continuable over a cut of singularities. Nevertheless, a major flaw was represented by entry 6, to which Painlevé addressed some critics later in 1902 ([75], p. 15):

> *"Borel himself had the clever and daring idea to use the series (M)*[16] *to extend the analytic functions over the singular lines and he gave some examples where this extension looks like natural. It's clear that one such idea can be acceptable only if it is not in contradiction with the classic analytic continuation. [...] This does not mean that idea should be dropped but that it can be followed only at the cost of imposing some restrictions to (M)."*

Painlevé advised that the star, although being a promising idea, could fail to work in the neighborhood of the poles of functions being meromorphic on the entire $\mathbb{C}$ for analytic reasons: he gave a counter-example ([76]) of a function including a point where all its n-th derivatives vanish without it to be identically zero: thus the sequence of series $f_n(x_0) \not\to F(x_0)$, failing to be uniformly convergent.

Borel was not late to reply to Painlevé ([19], p.151):

> *"These conditions*[17] *are not verified throughout the examples given by Painlevé. Thus such examples cannot bring any new restriction to the functions (M). [...] But let's get back to the so-called generalization of the analytic continuation: it is not required to be in contradiction with the classic theory, it's necessary that it shall not be in contradiction with itself and, most important, not too much complicated."*

Pailevé drastically concluded: '*It does not seems that, such a remarkable theory by Borel, could get out from the solely speculative terms until the day one could be able to construct explicitly a series* (M) *whose convergence enjoys the conditions stated by Borel, as it happens for Taylor series*'. It was not a surprise that even this very adaptive method cannot play everywhere to Borel anyway:

> *"Non-uniform convergence could bring in serious difficulties, therefore it could be preferable, although the relevance and the beauty of the results, to get contented to those series whose convergence inside any region would involve the uniform convergence inside all the region itself."*

Just to make some balance to Painlevé's attacks, we anticipate that Helge von Koch gave a contextual application of the star-like tool shortly later this episode (in 1906): we reported it at p. 21 with some computer graphics support.

6. Tracing the (r)evolution: Borel vs. Poincaré

In 1895 ([7], p. 47), strong of his results, Borel felt to blame blazoned analysts of deliberately ignoring relevant questions arising from monogenic non-analytic functions. Indeed, those 'antagonists' included integralists protecting the concepts of Continuity and of Differentiability from any disruption and those defending the 'infallibility' of Weierstrass' analytic continuation. Such straight and still positions sounded like religious dogmas, clashing with the open-minded, restless spirit of Science.

Poincaré belonged to the first party. In [82] (pp. 158-160), he labelled the new discoveries as 'bizarre' and 'strange'. His call for a sober spirit, more oriented to

[16] According to Borel's definition of series working with the star-like model.

[17] The ones we also listed before

practice than to abstraction, revealed that he was afraid that the whole corpus of Mathematics could have been destabilized, as it is evident from these laconic words:

> *"The building of a new function is demanded for goals of practical sense; up to present, new functions are putting the reasoning of our fathers in fault."*

What was behind this forthcoming crisis in Analysis ?

Poincaré was right the examiner of Borel's thesis and did not miss to welcome the new, promising results with fair felicitations [81]. Anyway, comparing Poincaré's opinions [80] to Borel's [7], we notice frictions originating from Borel's attempts to find functions which could be continued beyond their essential boundaries $\mathcal{L}$: one explicit consequence from the star-like model (see p. 8). When Poincaré in 1892 introduced the concept of lacunary series (see here p. 7), he stated that no function can be continuable over $\mathcal{L}$, believing this was the solely case. One natural critics came up when Borel found a counter-example, also similar to that Poincaré's configuration. Moreover, Poincaré was not an estimator of Cantor's results, one of the basis of Borel's approach ([84] and [26], p. 266).

Events are as complicate here as their matters and it is not surely fair to re-frame them into the right/wrong duality. Relying upon Weierstrass' continuation, Poincaré's results were *contextually correct* ([7], p. 19):

> *"Two analytic functions, coinciding in one region of the plane, coincide all over it or, at least, in all the regions which can be reached via analytic continuation."*

About that, Borel remarked ([7], pp. 10, 19):

> *"From it, Poincaré concludes that the notion of analytic continuation of a function outside a space bounded by a singular essential line is necessarily deprived of sense. [...] I showed how the difficulties pointed out by Poincaré are related to the definition being usually given to the monodromy of functions."*

Aware of discrepancy, Borel anticipated the forthcoming crisis of concepts ([7], p. 19):

> *"But given the case where a function has a closed essential singular line, we do not know what one can intend as analytic continuation beyond that line. [...] I took the question back from a different viewpoint and I showed that, in some cases, it is possible to give a definition of analytic continuation, beyond a closed essential singular line, which is not in contradiction with itself or with the previous one."*

Finally we account this episode, as told by Borel himself, epitomic of the reaction coming from one among the most famous representatives of the previous generation of analysts, educated to the reverence to monster theorists, such as Cauchy, Hermite and Weierstrass were considered ([20] (p. VII)):

> *"I will always remember the astonishment which I noticed in Mittag-Leffler, to whom I tried to explain the research projects, making no efforts to enter my thoughts and be content to pull a Weierstrass' memoir out from his bag and show me a sentence which should have completely close any discussion:* Magister Dixit.*"*

Meditation Within a general skepticism, Borel's ideas signed the times and explicitly opened to the study of non-analytic functions ([7], p. 47), for which it was relatively easy to build up power-series with an essential line of singularities: this geometric object resumes one peculiar property of most Julia sets.

7. Borel the moderator

Mathematics is the Art of making models. And none is up-to-date until it works fine. That was not Borel thought for the Series Theory indeed ([18], p. 1117):

> *"Consequently one can build up a theory including Weierstrass' theory as a special case and which is surely more general, since one actually knows new cases to which it applies."*

His position are very clear, lucid and fair: there exist monogenic functions, non-analytic also, that cannot be studied through the classic tools, such as differentiation. Just because they do not enjoy that property anymore. But this does not infer that the rest of the theory shall be dropped ([21], p. 80): Borel meant that there were important flaws in the theory, asking to be fixed. One was about *the general classification of functions*. Are there more families than known? If so, how many?

Another weak point was represented by the convergence tests: Bertrand's and Cauchy's, widely applied, are based upon limits. Both creak when they return 1; here, a specific approach is required to understand '*on how the limit* $\sqrt[n]{|c_n|}$ *tends to* 1' ([16], pp. 12). Borel lamented that there exist lots of cases where these limit-based tests *fail* or they are *inapplicable*, because of ambiguities. Fortunately, such cases often come up for series built *ad hoc* and very rarely from real world accidents; in any case, they can be decomposed into a bunch of simpler and tractable series. Borel challenged geometers to list related examples, if any (pp. 12–13). He concluded that Bertrand's and Cauchy's test are already sufficient for most of series arising from practice (p. 14).

So series showed a bunch of concepts which frame as a solid body for real world situations, but as a poor equipment to venture into the realm of abstraction and of generalization.

Getting conscious of the on-going trend devoted to examine most abstract power series, he presaged that the theory was going to lose efficacy. Borel accounted in 1898 ([14], p. 1001) that more and more researchers were joining the cause. We mention Fabry [31, 32, 33], taking up Hadamard's open questions, Lindelöf, attacking the question through conformal maps ([64, 65]), Leau [59] and Le Roy [60, 61, 62]), who independently worked on Borel's theory of divergent series.

According to the following, Borel warned that a global understanding of series was not at hand in late 1890s; hence contemporary analysis was restricted to classes of series which, however wide, were still calling for new approaches and theorems ([11], p. 1052):

> *"The importance of Weierstrass' definition is mostly due to the uniqueness, when existing, of continuation; but there are cases where it is not, for example, if the function has a natural boundary.*[18] *It seems hardly possible to hope to define, in all cases, a continuation satisfying such conditions. But it is not required at all to generalize the definition of Weierstrass; it suffices to yield a definition of continuation which holds in a large number of cases than those of Weierstrass and still enjoying*

[18] If singularities are so dense that no path can be traced without meeting one of them on the boundary, then the power series is said to have a 'natural boundary'.

> *the same remarkable property. Moreover there will be cases where the new definition will not permit any continuation; here one will continue to state that the function has a natural boundary until a new extension of the concept of continuation allows us to overcome this limit. However each new extension will not surely yield a unique result, that is, not being in contradiction with itself but also not in contradiction with the previously given definitions."*

Basing his evidence on current methods, Borel did not believe a generalization could be afforded; rather he like to reframe the concept of singularity and of continuation, depending on the nature of the given cases ([15], p. 284).

Also in Germany showed, there was a larger and larger opinion that Weierstrass' theory installation was appearing obsolescent. Borel pushed to a systematic study of non-analytic functions for which the classic approach via series expansion shows to be insufficient or inadequate. But he understood that he had to backtrack, because lacking of adequate tools. So he softened tones, preventing to be set at the head of the new trend, as readers might get out from the number of quotations to his works (for example, by Fabry, Leau, Le Roy), as the following one ([34], p. 78):

> *"When an analytic function has a singular closed line, one considers such a function as making no sense anymore if the variable crosses this singular line. In effect the analytic continuation by Taylor series is impossible. Borel was successful to give, in some cases, a rational definition of the analytic continuation* (Annales de l'École Normale, 1895)*, representing the function via an expression converging on both sides of the singular line. I intend to show that one can generalize, through only the consideration of continuity, the definition of analytic functions in order to continue them, for very general cases, beyond singular lines."*

One will see how far this spirit was from Borel's goals. There were two reasons for him to do so. First (p. 47) he was few (not completely!) close to the sense of Poincaré's attacks anyway and he alluded to the infinite chances offered by Analysis which, due to his character of being a Science operating with conceptual entities rather than material like Physics for example, has an infinite range of expansion and freedom of generating and working with objects of greater and greater generality (p. 49), that is, regardless of keeping particular properties:

> *"One can also state that non-analytic functions enjoy few common properties as well because, given a property, a clever analyst will often know how to build up a function which does not enjoy it; this is how one finds continuous functions with no derivative, or admitting the derivative at rational values of the variable only."*

Borel would come back later again in 1900 ([17], p. 1063):

> *"One can achieve wider generalizations by making no restrictions to the degrees of $R_n(z)$ [19], nor on the possibility for the same pole to appear in*

[19] After showing that the results for series of rational functions

$$\sum_{n=0}^{\infty} \frac{A_n}{z - a_n} \tag{7.1}$$

investigated by Poincaré [80] initially in 1892 and again in [83] could be extended to series of the more general form

$$R_n(z) = \sum_{n=0}^{\infty} \frac{P_n(z)}{Q_n(z)}. \tag{7.2}$$

> *infinitely many denominators; but the conditions of convergence which are required to impose to on the coefficients are no longer independent from the denominators, which is a great complication for applications."*

Secondly, non-analytic functions may have a good impact on non-analysts too (p. 48), so wishing for his call to be useful elsewhere in Science:

> *"Moreover these different functions could not rouse the interests of analysts exclusively. It seems that they might play a remarkable role in Physics as well, where they have not been introduced up to now simply because they were unknown (or, at least, known a few)."*

In this direction, Borel mentioned (pp. 48-49) a couple of examples of functions occurring in Physics (like refraction index and molecular mass distribution), to which series do not fit as solutions (p. 49):

> *"It is well intended that one can apply arbitrary approximation formulas; but often one tries to draw theoretical consequences from the expression of a formula, been found by gratuitously assuming that Taylor series applies to a function in Physics. However it is easy to realize how, on the contrary, it is few probable that Taylor formulas can be applied to the functions occurring in Physics."*

This is his honest, moderate position of his mathematical disappointments (pp. 47-48):

> *"But the serious inconvenient of such very general representation methods is that, when the function enjoys a simple property, this does not show up in the expansion; it suffices to cite the development in x as trigonometric series."*

Borel lamented that lifting the coefficients to highest abstraction deprived the series from the minimal support for working out its properties. Analysis was thus almost impracticable ([7], pp. 24 and 30):

> *"One realizes how difficult it is to study these functions in general terms, that is, without supposing anything on the distribution of the poles."*

These were the two complementary parts of Borel's train of ideas, driven by his strong empirical spirit and promoting what he firmly believed: a great theory is the one being applicable to real world (p. 50):

> *"... but the essential, in Mathematical Physics, is that one can build up a bunch of mathematical theories sharing, with a bunch of phenomena, a certain number of analogies thanks to which one could discover other ones. This is enough for a theory to be useful."*

And he definitely closed this point by warning Analysts to stick around a certain modern approach (far from series where the singularities distribution pattern could be known, so that the domain of convergence was also easy to be detected), if they are keen on making consistent researches:

> *"... but it is clear that, by simply transforming the fundamental condition, i.e. the remainder of the limited Taylor series shall tend to 0, without giving a precise analytic definition of the function, one would only reach to trivial tautologies."*

Also Hadamard shared such similar feelings to Borel's on the fate of the theory of power series. Despite Borel's achievements on power series in the end of XIXth century, the goal of a general theory proved elusive and remained so during the next 20 years.

It was clear that the generalization resumed into the definition of the domain of convergence and the nature of its boundary for any series, but (1) *the collected results gave no hope that a it was at hand* and (2) *the methods employed were* ad hoc *in the sense that were rooted to a particular example.* In a short communication of 1896, summarizing his latest results, Borel acknowledged that the theory was weak, in that it said nothing about 'unknown' series ([11], p. 1052):

> *"These results* [20] *describe some very interesting ideas developed by Fabry in the* Annales de l'École Normale [31]*; moreover they prove that from now on it makes sense to pose the following problem, whose statement shows that any* a priori *attempt to set the power series as a basis["as a basis" perhaps not right?] of analysis is at [the very] least premature: given a power series what determined]specific?] conditions must be satisfied by the coefficients so that the circle of convergence is not a cut?"*

Le Roy echoed Borel in his concern about the effective power to apply results for generalized series to concrete examples and also gave a very lucid vision of the theory current direction, opening his paper from 1899 as follows ([62], p. 492):

> *"An entire series with a circle of convergence defines an analytic function all over its domain of existence. But such a definition remains merely theoretical if one has no means to recognize, except through investigation of the series itself, whether continuation is possible or not. I endeavored to fill this lack[void? whole? gap?] by looking for the characters[means] to discover the nature of a given function by its power series expansion."*

Le Roy was very clear: advancements were solely possible at the cost of dropping the reverence to the old theoretical establishment ([63], p. 318):

> *"One knows that the only knowledge of the series still allows, at least theoretically, to define with no ambiguity the function at all points of its natural domain of existence. In other terms, if one adopts the notion of analytic function such as Weierstrass gave it, it is possible – as Poincaré showed – to yield, starting from the investigated series as initial element, a regular sequence of operations generating a countable set and allowing to approach the function at any point where it is holomorphic.*
>
> *However, one does not keep any general method to know, by means of a simple inspection of the series, if it is continuable, nor, in the case it is, to see, within the series itself, some indications of the behavior of the function at each point where it exists. From this, it comes that if the power series expansion is a good theoretical definition of the function, it is not a good practical definition."*

8. These doors have been made for opening

Beyond Taylor family, the series of rational fractions was seen as the immediate step ahead to larger generalization of algebraic formulas ([7], p.12):

> *"The first analytic expressions coming up after the algebraic and the rational functions are series of rational fractions which include, as special case, the series of polynomials."*

[20] On the condition of the coefficients for the circle of convergence to be a cut.

Borel's early memoir, together with alia, invited to drop the home comforts and start a journey towards new venues, with risks and mysteries. Not too late, in 1898, one Painlevè's theorem stated ([74], p. 201):

> *"Each function $F(z)$, single-valued in one natural domain of existence, can be represented by a series of rational fractions*
>
> $$F(z) = \sum R_n(z),$$
>
> *where the series is absolutely and uniformly convergent in any region of the plane where F(z) is holomorphic."*

Painlevé also remarked: *"If the set E* [of singularities] *encloses some lacunary spaces, one can lead the expansion $F(z) = \sum R_n(z)$* ad libitum *so that it diverges in these lacunary spaces or that it represents one arbitrarily chosen function in each one of these spaces."*

The theorem opens to a number of topological configurations for the domains of convergence of a given function. Borel deduced dangerous aftermaths for classic series theory ([17], p. 1061):

> *"Their*[21] *relevance in the general theory of functions must, due to this fact, seem considerable. But it is particularly diminished by the following statement, given in its most general form by Painlevé:* Given any number of domains, with no common parts [see figure 6], inside each a single-valued analytic map is defined; one can generate a series of rational fractions representing each function in the corresponding domain [...] *However the natural domain of existence of each function can be extended beyond the domain in which it is represented by the given series (of the series it represents?); but there is no longer any relations between the function and the series.* [...] *There is no necessary relation between the poles of the series and the singularities of the function which represents. Therefore this mode of representing functions appears very fallacious, precisely due to its too much generality (over generality? excessive generality?)."*

Painlevé's theorem inferred that a one-to-one connection cannot be always set up between the function, the series representing it and the so called 'domain W of Weierstrass'[22] because there might be functions whose total domain consists of disjoint regions as in figure (6). Mathematicians could have trusted that this was the solely case. Maybe Weierstrass, comforted by the examples he found, did it so while elaborating his continuation method.

One 'drawback' from Painlevé's statement was that multiply connected domains *were calling for a global scan over* $\hat{\mathbb{C}}$. The breaking news was that any continuation (Weiestrass', Mittag-Leffler's) approach was insufficient to detect the whole domain of the function, in fact it just works inside one component exclusively. Starting from one component, how could one have reached to the other ones, being not right over the boundary ? And how many would have they been?

In 1919–20 Pierre Fatou (1878–1929), in the course of his studies on the iterates of rational functions ([35], p. 88, IV), accidentally showed an example of such basins topology (here depicted in 6/B).

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[21] Series of rational fractions.

[22] The one and only connected open region which is extendible by standard analytic continuation via function elements.

(A) (C)

Figure 6. (A) Sketch of the domain of existence for a rational map, according to Painlevé's theorem. (B) The basin of attraction to the finite fixed point for the iterates of $z^3 + 3z^2 + 2z$ (Fatou's example).

9. Opening to the new century

When XXth century opened, the questioning on series was still hot. Events swang for years between opposition and favor, prejudices and enthusiasm. Borel's concerns looked like Darboux's twenty years later[23] and were perhaps part of a deeper misgivings shared by other, including Picard[24], towards the directions taken by new researchers, where generality, axiomatization, abstraction were exaggerated, according to Borel's and Lebegue's opinions. As often happens, Borel and Lebesgue were on the other side of this dispute when they were younger. Lebesgue, after all, was known as the perpetrator of pathological functions without derivatives; whereas Borel, in 1895, found himself playing the role of (soft) antagonist against the rigidity shown by some French establishment in favor of Weierstrass' results,

[23]Borel and Darboux, in different times and contexts, were not completely satisfied (even worried somehow) about the recent events in Mathematics (read Analysis): in both cases, the sought robust stability of this field itself was menaced, either in terms of lacking rigor (Darboux) or when (Borel) series theory got to an high level of abstraction so that application to phenomena was at risk: many new theorems were holding at a formal level exclusively.

[24]Picard felt disturbed by the recent paths followed by the new generation of French analysts and, in the occasion of the reviewing of Lebesgue's thesis in 1902 and although taking it in a very high consideration, he wrote as follows ([40], p. 384):

> *"One knows how much the concept of function widened since twenty years, and how functions, enjoying the most strange properties, have been discovered; sometimes one goes to one's head while looking up at the approached results as the usual hypotheses are given up. Those transported to such speculations, they need a great abstraction power and shall mistrust the intuition.* [...]
>
> *Lebesgue's cast of mind carries him naturally to the study of questions of principle worrying a number of geometers up to now; however he is not an intransigent, and one does not find the mistrust of intuition in him, becoming a craziness among some of our contemporaries.* [...]*"*

in his advocation of more general ways of looking at continuation, as suggested in this remark by Borel in 1917 ([20], p. VI):

> *"Poincaré generated, with great talent, some analytic expressions presenting singularities and believed to be able to infer and conclude the impossibility of extending the theory of analytic functions beyond the limits fixed by Weierstrass* [...]*"*
>
> *"In my Thesis* [6]*, for which Poincaré gave me the honor to be its referee, I refused this negative viewpoint."*

Despite of controversies and the precarious overall status, the final balance was that the XIXth century closed with a significant progress.

Inspired by the behavior of Cauchy's integral (3.1), which assumes distinct values in distinct regions, Borel settled against the results by some mathematical gurus by explicitly questioning about the splitting role of the boundary between disjoint domains of convergence ([7], p. 9):

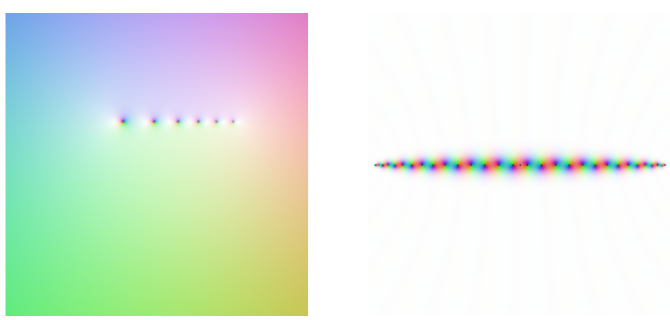
> *"Given two functions in one complex variables where one function is defined in a certain domain and the other inside a different domain, when can state that it is the same function ?"*

Or, switching to the domains of convergence, this same question could also be reverted ([7], p. 15):

> *"Given two areas such as S' separated by some lines or by areas of singular points, can they be related to two analytic functions defined by the series* [$\varphi(z)$] *inside these two areas so that one is the continuation of the other?"*

Borel started his discussion by joking with obviousness (p. 9):

> *"... one gets the same function when one has the same analytic expression."*



(A) (B)

FIGURE 7. (A) The essential cut for Borel's example of the series (9.1); (B) An analogous Julia set arising from the iterates of $f(z) = z^2 - 1$. In both cases, we have just one boundary line. Being not a closed curve, there exists only one domain of convergence

In the economy of a theoretical development, Borel's words tell more than just solving a question of convergence. Some preparatory concepts allowed him to state: *if an algebraic relation between $\varphi(z)$ and its derivatives holds true in any region of* *the plane, it is an identity and thus holds all over the plane.* Borel came to these considerations ([7], p. 18):

> *"the preceding lines are enough to legitimate the following convention: we will state that the function*[25] $\varphi(z)$ *represents the functions at all points where it is convergent; thus we will introduce a certain class of functions enjoying perfectly determined properties and not keeping any contradiction point. But this definition might, without being in contradiction with itself, be in contradiction with the ordinary notion of analytic continuation – which would be a serious inconvenient."*

In the course of his early work, he proved ([7], p. 10) the existence of cases where continuation can be afforded without falling into contradiction. Assuming that a sufficient basis of properties was known about the given series, Borel worked with this configuration: given a line $\mathcal{L}$, let S be a perfect set of isolated poles, distributed on $\mathcal{L}$ or not, and let S' be the set of singularities or of limits of S: for those elements of S and inside $\mathcal{L}$, the derived set S' is again on $\mathcal{L}$, otherwise, if $a' \in S'$ and $a \notin \mathcal{L}$ is assumed, the limits a' are isolated. Borel also assumed that such lines $\mathcal{L}$ are tangent provided. It was not a pure artifice for a proof, but rather was the configuration for many series endowed with a singular or (specifically) an essential line. Borel extended this topological configuration and related conclusions to the cases when $\mathcal{L}$ is not closed as well.

Borel gave ([7], p. 13) examples of new series with unusual features, like this one whose singularities are poles distributed along a straight-line (see figs 7):

$$\sum_{n=1}^{\infty} \frac{1}{z - \sqrt{n}}. \tag{9.1}$$

After the old school (Mittag-Leffler, Picard, Poincaré), after Borel standing in the middle, we introduce two major figures – Fabry and Le Roy –, whose contributions will help to figure out next events.

Meditation During late 1890s, Borel worked on configurations pretty close to Julia sets: closed and perfect sets, with the exception that they are tangent-provided here.[26] As shown at p. 9, the existence of derivatives of any order - and thus the uniform convergence - is a requisite that Borel could not drop, or its construction would have failed.

10. 20th century boys

With a similar opinion as Borel's, Le Roy (1870–1954) seemed to be already aware that some questions regarding the application of series theory to any case (general) would have probably failed. Le Roy noticed that it could always score the goal of generalizations, but in other meanings. His start points were Borel's developments, in particular one theorem (ref. [13]) about entire series whose domain of convergence is bounded by a singular line ([63], p. 319):

---

[25]Borel specified earlier that it can be expressed as follows:

$$\varphi(z) = \theta(z) + R(z) \qquad \theta(z) = \sum_{n=1}^{k} \frac{B_n}{(z-\alpha)^n}, \qquad R(z) = \sum_{n=k+1}^{\infty} \frac{B_n}{(z-\alpha)^n}.$$

[26]This property is peculiar to the simplest case of Julia sets arising from the iterates of the power family $f(z) \equiv z^n$.

> *"Borel's theorem relates to arbitrary series whose coefficients are randomly given. But the most general function does not correspond to the most general series. From the ordinary viewpoint of Analysis, the most general function is actually the one for which the distribution of singularities is not subjected to any rule. Therefore it is assured that, if one is bounded to the natural questions in Analysis, where the series to be considered are brought by calculus and not a priori posed, the most frequent and useful functions will be the ones which can be continuable."*

Le Roy was sure that he could made further advancements to Borel's conclusions:

> *"I trust that I can deduce, from above, trying mostly to define classes of continuable series, I will not bounded the practical power of the researches I am going to take up. Therefore I will specially refer to the discovery of the continuability features. There is a theory to build up, being similar – again – to the ones regarding the convergence of series."*

He started from the Taylorian series ([63], p. 350)

$$f(z) = \sum_{0}^{\infty} \alpha_z z^n \qquad \text{with} \qquad \alpha_n = \sum_{p=0}^{\infty} \frac{\lambda_p}{p!} n^p. \tag{10.1}$$

And showed that the domain of convergence can be more easily investigated by means of Cauchy-like integrals (3.1), evaluated on the boundaries, rather than the study of the series itself. We noticed that (ref. to [63], pp. 428-429 and more concisely to [63], p. 337) integrals themselves are subjected to the choice of the best one defining the given series:

> *"The method I proposed is based upon the application of some integrals in one variable z. In all cases, the problem of the analytic continuation refers back to the choice of a simple and appropriate method for the factor in the differential coefficient containing z, because this is a factor determining the properties related to the continuity of the integral and the possibility to write the coefficients of the given series under the form of some definite integrals, because this allows to identify the series in question with an integral of chosen kind."*

Playing with parameters in order to classify series whose coefficients α_p obey to given rules, Le Roy pointed to the existence of domains of holomorphy[27] whose boundary might not be essential and, mostly, be no more a topological circle, so that different shapes may arise. For example, one can find cases where it can be a curve, parametrized by ([63], p. 350):

$$x = e^{-\frac{\cos\theta}{a}} \cos\left(\frac{\sin\theta}{a}\right), \qquad y = e^{-\frac{\cos\theta}{a}} \sin\left(\frac{\sin\theta}{a}\right),$$

where $-\pi < \theta < +\pi$, $a_n = n^p/p!, p \in \mathbb{N}$.

With a same flavor, the young Fabry got into this topic during 1896. Like Le Roy, he also begun from the latest Borel's results and. In one early long work

[27] One can deduce that, exclusively inside $\mathbb{C}^1$, every open set K is a domain of holomorphy when one can define a holomorphic function f whose zeros accumulate on its boundary and thus f cannot be continued outside K; one example is in figure (4). This condition refers to derivability of f: it relates to the possibility of defining the region of definition for the series representing f and, on the contrary, of detecting the singularities (where no finite derivative can be computed, such as at poles or essential points). In $\mathbb{C}^{n>1}$, this values distribution, characterizing the domain of holomorphy, does no longer hold.

[33], Fabry focused on the series (10.1) and assumed (§2) the Cauchy convergence criterion (a) $\sqrt[n]{|\alpha_n|}$ into the logarithmic version: (b) $\Psi(\alpha_n) = \frac{\log|\alpha_n|}{n}$; as $(a) \to 1$ for $n \to \infty$, (b) tends to 0. This tool allows a more comfortable work with exponents of coefficients as (§5), given $\alpha_n = \rho_n e^{\omega_n i}$:

$$\sum_{p=0}^{\infty} z^p \rho_n e^{\omega_n i}; \tag{10.2}$$

He started from a simpler form of (10.2) and found a theorem (p. 78), where singularities spread on the unit circle. Fabry then moved from the unit disc to a trigonometric series, where ω is replaced by a function:

$$f(z) = \sum_{p=0}^{\infty} z^p \rho_n e^{in\varphi(n)}. \tag{10.3}$$

With the goal of finding the boundary of its domain D of holomorphy, Fabry stated that $\Psi(\rho_n) \to 0$ (or $\sqrt[n]{|\alpha_n|} \to 1$), for any sequence of values n and he also showed that one can construct sequences of arcs completely included inside D and whose limits, as $n \to \infty$, are lines of singularities along the boundary ∂D, that is, f is no more defined beyond ∂D. Remarking their role, Fabry wrote ([32], p. 78):

> *"And, if such limits give rise to a continuous sequence, one will find out arcs, of the circle of convergence, acting as cuts."*

Fabry did not apply the analytic continuation but a strategy with a vague 'dynamical' flavor, considering the sequences of curvilinear arcs and showing that their limits coincide with curves whose points are right the same bounding singularities as sought for $f(z)$, so that

$$\limsup_{n=\infty} \partial D_n \equiv \partial D.$$

Fabry's novelty relied on the reasons why he was led to use more general geometrical entities than the Cauchy's radius of convergence, working when domains of convergence are discs but obsolescent for differently shaped regions. The above quotation evinces that, operating over $\widehat{\mathbb{C}}$, boundaries cannot be just thought as closed curves but as curvilinear segments, as Fabry ([32], p. 79) pointed out by turning the series (10.3) into

$$\sum_{n=0}^{\infty} z^n \rho_n e^{ian \sin(\log(n)^{\theta})}.$$

Fabry showed an example of how these curves, generated by series, evolve:

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+a)^p}. \tag{10.4}$$

The reader can interpret as follows: given

$$f_n = \sum_{n=0}^{\infty} \alpha_n z^n,$$

then each f_{n+m} $(m \in \mathbb{Z}, m \neq 0)$ shows up as a new traditional series g, together with its features (domain of convergence, singularities distribution, ...), so that one can reasonably write $g = f_{n+1}$ and this is why speaking of domain of holomorphy (because of its relation with the extension process by continuation) makes more sense in this new 'dynamical' context of f_n, which actually behaves as *catalogue of other series*, namely the standard one and each with its own domain of convergence.

Meditation From Le Roy's and Fabry's productions, it is clear that the path followed to extend the series theory is competing (a) the trigonometric series and (b) a new definition of coefficients, different from the Taylorian form and possibly relying upon auxiliary functions, such as the simple transcendentals.

In this new context, new works flourished, such as renown Fatou's thesis in 1906, largely credited for its avant-guard results[28] and being among the first ones merging Lebesgue's Measure Theory to Complex Analysis. Below an excerpt from Painlevé's review of Fatou's thesis ([40], p. 397):

> *"Fatou's work is mostly devoted to the study of the properties of a Taylor series on the circle of convergence and, as a consequence, to the study of the trigonometric series. These topics, of capital importance, gave rise to several works, in particular during the latest years: the last hidden results are now well-known and, if one realizes that the theory is not achieved and perfected yet, one also understands very quickly that the points left to be cleared raise very big difficulties."*

We also mention some early von Koch's production (yes, the same one as of the famous fractal curve) in the same year and about power series with unusual domain [93]:

$$-\frac{1}{n}\sum_{\nu=1}^{\infty}\frac{(1-z^n)^\nu}{\nu}. \tag{10.5}$$

For $\nu = 2$, (10.5) returns a series whose boundary of convergence is a lemniscate of Bernoulli, arising from a clever use of the transcendental *log* function. For the $\nu-$family, the boundary of the domain of convergence is not a simply connected curve, but intersects itself (see (8). The domain D of convergence is the union of the interior of the 'petals'. Notice that the Mittag-Leffler's star does fit one such topological configuration (see p. 8); on the contrary, Weiestrass continuation would be misleading about the origin. As von Koch remarked, such a star might be set up at the origin, so that the branches depart from it and cross each petal.

Meditation The boundaries of the domains of convergence for series may present more complicate topologies than the simple smooth curve. But constructing such examples via series is impracticable.

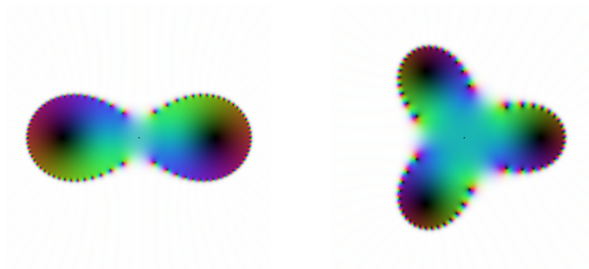
11. Keep it in the family

The function (5.1) is a rational map, better rewritten in this simpler Laurent polynomial:

$$F(z) \equiv \sum_{n=-\infty}^{+\infty} A_n(z-a)^n, P_n = A_n(z-a)^n \tag{11.1}$$

For definition, the series on the left converges to a value u_z assumed by the function F at z. So we have a convergent sequence of values $P_n(z) = u_n \to u_z \equiv F(z)$, or $\lim P_n(z) = F(z)$. In most cases, it is proven that this sequence is uniform. We start from a neighborhood of a point z, regular for P_n. Now let the subinterval

[28]Such as the celebrated theorem about the existence of almost everywhere radial limits for an holomorphic function $f : \mathbb{D} \to \mathbb{C}$



(A) (B)

FIGURE 8. (A) Series (10.5) in progress for $n = 2$ (B) Here for $n = 3$. The number of steps s is always 40. As $s \to \infty$, the 'petals' come to join at the origin. The black regions indicate that values are very close or equal to 0.

$1 \le n \le \infty$, so that u_n turns into the sequence of values assumed by $P_n(z)$, ([70], p. 29), arising progressively by summing up term to term inside (11.1) as n grows: $u_n = P_n(z) \equiv \sum f_n(z)$. The sequence can be reviewed as the sum of polynomials (or of meromorphic or entire maps) 11.1. Irregular points of this series could be critical, poles or essential singularities (p. 102). In general we assume that the P_n are analytic.

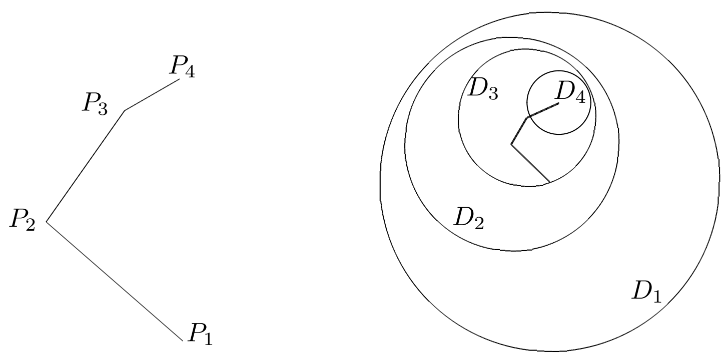


FIGURE 9. (A) Orbit generated by the successive approximations $P_n(z)$ of $F(z)$. (B) The same orbit reframed in terms of compact subsets $D_{n+1} \subset D_n$, where each P_n is the center of the disk D_n.

One such consideration inspired Paul Montel to revisit series in terms of sequence of functions, of which the series expansion is thus a subfamily. Synapses freak out when we take into account the following Montel's statement:

In a neighborhood of x_0, the equation

$$P_n(z) = a \tag{11.2}$$

has, starting from a given n, one and only one solution.

It immediately follows that, given $\Gamma(z) \equiv P_n(z) - a$ is univalent and invertible inside an arbitrarily close neighborhood D of a. If a is z, we have $P_n(z) = z$ and we can reasonably apply sequences of polynomials to the neighborhoods of fixed points γ, when 11.2 turns into this special form $P_n(z) = z$. If $\Gamma(z)$ is linearizable around γ along Schöder's equation terms:

$$\Phi(F(z)) = aF(z),$$

we can refer back to Koenigs' results on the local iterations in the neighborhood of an attracting fixed point. Here opens a new story (see [1]), for which this narration could be considered as just the prologue.

12. Conclusions

We mimic Borel, Fabry, Le Roy and lift our viewpoints, running the events again from a general overview and concluding that global holomorphic dynamics were the natural response to long time quests in Complex Analysis. It did not come up at random, just to fill the announce text of one French mathematical contest edition in 1915: one such hypothesis would not be consistent, because most of its organizers authored remarkable results and aware of the frontier questions. The concourse was banned for stimulating the circuit to elaborate radically new strategies for problems to which current Analysis could just give bare responses.

It is crucial to remind the reader that series and integrals were not privileged tools of investigation, but just the ones being at hand during the first era of modern Mathematics. Back to then, the approximation of solutions to polynomial equations was the true interest. The geometry of the domains of convergence was a later question, taking shape inside Cauchy's and Weierstrass' production and welcoming the readers for updates, like it happens today in the open-source world of software development. Formulas became also functional to the study of geometries. Crisis arose when new series lose the contact with the ground and took off, flying to highest degrees of complication. The hostilities met during these circumstances are just natural accidents along the route that led from Series Theory 1.0 (Cauchy, Weierstrass), forward to 2.0 (Borel, Hadamard, Painlevè) and then to 2.1 (Fabry, Fatou, Le Roy, von Koch et alia). Almost 20 years later, such 'uncertain' status for series theory was still in the air, but the alert level was no longer red, probably because some leaders, like Darboux and Poincaré, passed away and others, like Painlevé and Borel[29], have been also absorbed by new tasks taking them away from such disputes and bringing towards (academic and political) managerial positions. For example, Borel's viewpoint appeared softened and he cautiously welcomed the recent achievements ([21], pp. 77 and 82):

> *"To confine ourselves at the theory of functions, both the study or particular functions and the general theory shall not be neglected; but one could cultivate the general theories for themselves or looking for some special applications. It is evident that who cultivates the general theories in themselves cannot hope to apply them to a function which will be never built up;* [...] *I do not question, of course, the utility of the general theories and of bold syntheses; they are often very fertile, always softening the spirit and guiding it to the applications of the constructive method; but, in my opinion, they are a mean, not an end in itself."*

Bieberbach sentenced later in 1921: '*But it seems that the general method of has not been reached yet*' ([3], p. 459).

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[29]Despite of Painlevé, Borel did not quit his research activities.

At this closing stage, cunning readers would not leave the question open: why, back to 1890s, did Borel either push for extensions but keep some distance from generalization goals later? Here his possible reply ([21], p. 81):

> *"One can judge my viewpoint to enjoy some despicable empiricism. It will surely be nobler to deal with all the questions and all the problems at once. Unfortunately, even when one is able to state general results, the particular difficulties come up as one actually intends to get to the details of the applications."*

The theory went further than Borel's ambitions: in late 1890s, when he worked on series with arbitrarily random coefficients, the solely goal was to show the possibility of new boundaries besides the circle. He did not intend to open a new research path, probably foreseeing it could not get to anything satisfactory. There are no trade-offs for his lucid visions, based upon a strong conviction to which we could just quote here as homage:

> *"The whole Mathematics owes his origins and most of his progress to the observation and experience; this origin shall not be ignored; setting a ditch between Mathematics and Reality is a severe pedagogical error, which seems to be insufficiently fulfilled in our secondary school, although the efforts from the inspirer of the programs*[30] *in 1902. One should not forget now that the proper goal of Mathematics is to gather the elements shared by different realities into abstraction, so to give rise to theories whose field of application is as large as possible; this viewpoint is not at the opposite of the other, but otherwise it is completed. ... "*

As we rewind the events, we see that the boundary problem came out again elsewhere in late 1870s [23], under the shape of new complex formulas subjected to the iterative process. Cayley's call remained unheard for years. The French start-up was promoted by Gabriel Koenigs (1858–1931), pushing some of his students (Auguste Grèvy and Leopold Leau) to join him and continue the local investigations.

Nothing weird here: if series were already presenting delicate management issues, iterates could not rely on the minimal theoretical support at all, assuming that the theorems by Arzelà and by Ascoli on equi-continuity were not stated yet. Geometry and Analysis were running at different speed: applying the analytic continuation to reach to the singularities spread by iterates all over $\hat{\mathbb{C}}$ was like using the coal for propelling a space craft and travelling from Earth to Antares.

In retrospect, the iterations of complex functions and the compositions via discontinuous groups of complex fractional transformations – powered by normal families, Riemann surfaces, Teichmüller spaces, quasi-conformal mappings – show up as big interstellar ships with warp-drive, capable to construct functions with the most complicate singularities distribution through a relatively few number of steps. Put it simply, $f^n(z)$ and Mob= $PSL(2, \mathbb{C})$ can build complex functions $g(z)$, whose level of complication cannot be reproduced by humans otherwise, that is, via *ad hoc* complex series or functions construction. Those crinkly curves could tilt the already flexible star-based method too.

Between 1915–1918 Fatou and Julia jumped over the space ship constructed by Paul Montel and ventured to the whole Riemann sphere. Then the far away

[30] Borel referred to Georges Leygues' reform, placing the scientific teaching at the same level as literal, and renewing the former too.

singularities turned from imperceptible dots to identifiable stars with mass and role, either tractable alone or all in their set as constellations.

Similarly to what was already known for singularities of power series (refer to p. **??**), they found that the basins of convergence arising from iterations were bounded by points distributed in a range of geometries, from totally disconnected sets, along curves of increasing complication, up to the whole Riemann sphere.

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